Ultraviolet divergences

Regularization and renormalization

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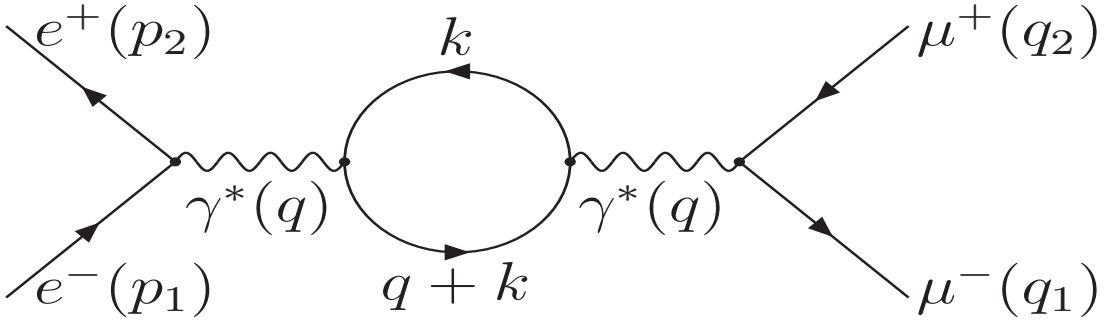
Virtual corrections

E.g. for the process

$$e^-(p_1) + e^+(p_2) \to \mu^-(q_1) + \mu^+(q_2)$$

Consider the Feynman diagram with the vacuum polarization contribution



The amplitude

$$M = \bar{v}(p_2) ie\gamma^\alpha u(p_1) iD_{F\alpha\mu}(q) ie^2 \Pi^{\mu\nu}(q) iD_{F\nu\beta}(q) \bar{u}(q_1) ie\gamma^\beta v(q_2)$$

contains the vacuum polarization tensor $\Pi^{\mu\nu}(q)$.

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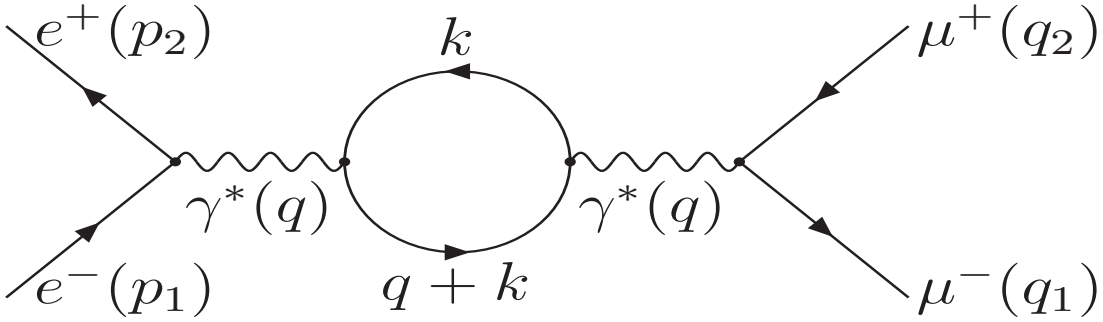

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contains the vacuum polarization tensor $\Pi^{\mu\nu}(q)$.

Vacuum polarization

$$
\begin{aligned}
ie^2\Pi^{\mu\nu}(q) &= -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[ie\gamma^\mu iS_F(k)ie\gamma^\nu iS_F(k+q)\right] \\
&= -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[ie\gamma^\mu i\frac{\not{k}+m}{k^2-m^2+i\varepsilon}ie\gamma^\nu i\frac{\not{k}+\not{q}+m}{(k+q)^2-m^2+i\varepsilon}\right] \\
&= -e^2\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu\left(\not{k}+m\right)\gamma^\nu\left(\not{k}+\not{q}+m\right)\right]}{\left(k^2-m^2+i\varepsilon\right)\left[(k+q)^2-m^2+i\varepsilon\right]}
\end{aligned}
$$

- If $|k^\mu| = \Lambda \to \infty$, then the the diagram is quadratically divergent. This kind of divergence is called the ultraviolet divergence.
- We must regularize the divergence.
- The best way to do it is the dimensional regularization.

Dimensional regularization

Let us define in the D-dimensional space:

$$
\begin{aligned}
g_{00} &= -g_{ii} = 1, \qquad i = 1, 2, ..., D-1, \\
g_{\alpha\beta} &= 0, \qquad \alpha \neq \beta, \\
k^{\alpha} &= \left(k^0, k^1, ..., k^{D-1}\right), \\
g_{\mu\nu} g^{\mu\nu} &= D, \\
\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\right) &= 4 g^{\mu\nu}, \\
\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right).
\end{aligned}
$$

Dimensional regularization

The basic integral in the D-dimensional space is:

$$I_0 = \int \frac{\mathrm{d}^D k}{(c^2 - 2k \cdot p - k^2)^n} = i\pi^{\frac{D}{2}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} \left(c^2 + p^2\right)^{\frac{D}{2} - n},$$

where the integration measure

$$\mathrm{d}^D k = \mathrm{d}k^0 \mathrm{d}k^1 ... \mathrm{d}k^{D-1},$$

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$$\Gamma(n+1) = n!, \qquad (n+1)! = (n+1)n!.$$

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$$\Gamma(x+1) = x\Gamma(x)$$

for real or complex arguments.

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Dimensional regularization

- The $\Gamma(x)$ function has poles at $x = 0, -1, -2, \ldots,$
- where

$$\Gamma\left(\frac{\eta}{2}\right) = \frac{2}{\eta} - \gamma_E + \mathcal{O}(\eta), \qquad \eta > 0,$$

- $\gamma_E = 0.5722\ldots$ is the Euler constant,
- The formula for I_0 is meaningful for any dimension D, in particular for

$$D = 4 - \eta,$$

- however, for $D = 4$ and $n = 2$ it is logarythmically divergent.

Dimensional regularization

- Tensor integrals in the D-dimensional space:

$$
\begin{aligned}
I_\mu &= \int \frac{\mathrm{d}^D k \; k_\mu}{(c^2 - 2k\cdot p - k^2)^n} = -p_\mu I_0, \\
I_{\mu\nu} &= \int \frac{\mathrm{d}^D k \; k_\mu k_\nu}{(c^2 - 2k\cdot p - k^2)^n} = \left[p_\mu p_\nu - \frac{1}{2} g_{\mu\nu} \frac{(c^2 + p^2)}{\left(n - 1 - \frac{D}{2}\right)} \right] I_0.
\end{aligned}
$$

- Excercise
  Prove the above formulae by differentiating I_0 with respect to $\partial/\partial p^\mu$.

Vacuum polarization

- We want to absorb the divergence of the vacuum polarization diagram in the electron charge. To this end let us define the bare charge of the electron e_0.
- In $D = 4 - \eta$ dimensions, we have

$$i e_0^2 \Pi^{\mu\nu}(q) = -e_0^2 \, \mu^\eta \int \frac{\mathrm{d}^{4-\eta} k}{(2\pi)^{4-\eta}} \frac{\mathrm{Tr}\left[\gamma^\mu \left(\not{k} + m\right) \gamma^\nu \left(\not{k} + \not{q} + m\right)\right]}{ab},$$

- where

$$a = k^2 - m^2 + i\varepsilon, \qquad b = (k+q)^2 - m^2 + i\varepsilon,$$

- and μ is an arbitrary mass parameter.

Feynman parametrization

- If there are two factors in the denominator, then we have:

$$\frac{1}{ab} = \frac{1}{ba} = \int_0^1 \frac{\mathrm{d}z}{[a + (b-a)z]^2}.$$

- If there are three factors in the denominator, then we have:

$$\begin{aligned} \frac{1}{abc} &= 2\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \frac{1}{[a + (b-a)x + (c-b)y]^3} \\ &= 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}z \frac{1}{[a + (b-a)x + (c-a)z]^3}. \end{aligned}$$

Both formulae can be easily proved by the direct calculation of the integral.

Feynman parametrization

- If there are $n+1$ factors in the denominator, then we have:

$$\frac{1}{a_0 a_1 \ldots a_n} = n! \int_0^1 \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 \ldots \int_0^{z_{n-1}} \mathrm{d}z_n \frac{1}{[a_0 + (a_1 - a_0)z_1 + \ldots + (a_n - a_{n-1})z_n]^{n+1}},$$

which can by proved by mathematical induction.

Vacuum polarization

For

$$a = k^2 - m^2 + i\varepsilon, \qquad b = (k+q)^2 - m^2 + i\varepsilon,$$

we have

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[a + (b-a)z]^2} = \int_0^1 \frac{\mathrm{d}z}{[m^2 - i\varepsilon - q^2 z - 2k\cdot qz - k^2]^2}.$$

Thus

$$i e_0^2 \Pi^{\mu\nu}(q) = -e_0^2\, \mu^\eta \int \frac{\mathrm{d}^{4-\eta}k}{(2\pi)^{4-\eta}} \frac{\mathrm{Tr}\left[\gamma^\mu (\not{k} + m)\gamma^\nu (\not{k} + \not{q} + m)\right]}{ab}$$

$$= -e_0^2 \left(\mu^2\right)^{\frac{\eta}{2}} \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^{4-\eta}k}{(2\pi)^{4-\eta}} \frac{\mathrm{Tr}\left[\gamma^\mu (\not{k} + m)\gamma^\nu (\not{k} + \not{q} + m)\right]}{[\underbrace{m^2 - i\varepsilon - q^2 z}_{c^2} - 2k\cdot \underbrace{qz}_{p} - k^2]^2}.$$

Vacuum polarization

For

$$a = k^2 - m^2 + i\varepsilon, \qquad b = (k+q)^2 - m^2 + i\varepsilon,$$

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Vacuum polarization

We see that

$$c^2 = m^2 - i\varepsilon - q^2 z, \quad \text{and} \quad p = qz.$$

Let us definne the function

$$D_2(z) \equiv c^2 + p^2 = m^2 - i\varepsilon - q^2 z(1-z).$$

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Let us calculate the trace in the numerator

$$\begin{aligned}
& \mathrm{Tr}\left[\gamma^\mu \left(\not{k} + m\right) \gamma^\nu \left(\not{k} + \not{q} + m\right)\right] \\
= \; & \left(k_\alpha k_\beta + k_\alpha q_\beta\right) \mathrm{Tr}\left(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta\right) + m^2 \mathrm{Tr}\left(\gamma^\mu \gamma^\nu\right) \\
= \; & \left(k_\alpha k_\beta + k_\alpha q_\beta\right) 4 \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\nu}\right) + m^2 4 g^{\mu\nu} \\
= \; & 4 \left[2 k^\mu k^\nu - g^{\mu\nu} k^2 + k^\mu q^\nu + q^\mu k^\nu - g^{\mu\nu} k \cdot q + m^2 g^{\mu\nu}\right].
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= \; & \left(k_\alpha k_\beta + k_\alpha q_\beta\right) 4 \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\nu}\right) + m^2 4 g^{\mu\nu} \\
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Vacuum polarization

Thus

$$
\begin{aligned}
& i e_0^2 \Pi^{\mu\nu}(q) \\
= \quad & -e_0^2 \left(\mu^2\right)^{\frac{\eta}{2}} \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^{4-\eta}k}{(2\pi)^{4-\eta}} \frac{\mathrm{Tr}\left[\gamma^\mu \left(\not{k} + m\right) \gamma^\nu \left(\not{k} + \not{q} + m\right)\right]}{\left(c^2 - 2k\cdot p - k^2\right)^2} \\
= \quad & -e_0^2 \left(\mu^2\right)^{\frac{\eta}{2}} \int_0^1 \mathrm{d}z \\
\times \quad & \int \frac{\mathrm{d}^{4-\eta}k}{(2\pi)^{4-\eta}} \frac{4\left[2k^\mu k^\nu - g^{\mu\nu}k^2 + k^\mu q^\nu + q^\mu k^\nu - g^{\mu\nu} k\cdot q + m^2 g^{\mu\nu}\right]}{\left(c^2 - 2k\cdot p - k^2\right)^2}
\end{aligned}
$$

Vacuum polarization

$$
\begin{aligned}
i e_0^2 \Pi^{\mu\nu}(q) &= -e_0^2 \, \frac{\left(\mu^2\right)^{\frac{\eta}{2}}}{(2\pi)^{4-\eta}} \, 4 \int\limits_0^1 \mathrm{d}z \\
\times \quad & \int \mathrm{d}^{4-\eta} k \, \frac{\left[2k^\mu k^\nu - g^{\mu\nu} k^2 + k^\mu q^\nu + q^\mu k^\nu - g^{\mu\nu} k \cdot q + m^2 g^{\mu\nu}\right]}{\left(c^2 - 2k \cdot p - k^2\right)^2} \\
= \quad & -\frac{e_0^2}{4\pi^4} \, \frac{1}{\left(4\pi^2 \mu^2\right)^{-\frac{\eta}{2}}} \int\limits_0^1 \mathrm{d}z \\
\times \quad & \left[2 I^{\mu\nu} - g^{\mu\nu} g_{\alpha\beta} I^{\alpha\beta} + I^\mu q^\nu + q^\mu I^\nu - g^{\mu\nu} q_\alpha I^\alpha + m^2 g^{\mu\nu} I_0\right]
\end{aligned}
$$

Vacuum polarization

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\begin{aligned}
i e_0^2 \Pi^{\mu\nu}(q) &= -\frac{e_0^2}{4\pi^4} \frac{1}{(4\pi^2\mu^2)^{-\frac{\eta}{2}}} \int_0^1 \mathrm{d}z \\
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= \quad & -\frac{e_0^2}{4\pi^4} \frac{1}{(4\pi^2\mu^2)^{-\frac{\eta}{2}}} \int_0^1 \mathrm{d}z \left[2\left(p^\mu p^\nu - \frac{1}{2} g^{\mu\nu} \frac{(c^2+p^2)}{\left(1-\frac{4-\eta}{2}\right)}\right) I_0 \right. \\
& -g^{\mu\nu} g_{\alpha\beta} \left(p^\alpha p^\beta - \frac{1}{2} g^{\alpha\beta} \frac{(c^2+p^2)}{\left(1-\frac{4-\eta}{2}\right)}\right) I_0 + \left(-p^\mu I_0\right) q^\nu \\
& \left. +q^\mu \left(-p^\nu I_0\right) - g^{\mu\nu} q_\alpha \left(-p^\alpha I_0\right) + m^2 g^{\mu\nu} I_0\right]
\end{aligned}
$$

Vacuum polarization

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\begin{aligned}
i e_0^2 \Pi^{\mu\nu}(q) = & -\frac{e_0^2}{4\pi^4} \frac{1}{(4\pi^2\mu^2)^{-\frac{\eta}{2}}} \int_0^1 \mathrm{d}z \\
\times \quad & \left[2\left(q^\mu q^\nu z^2 - \frac{1}{2} g^{\mu\nu} \frac{D_2(z)}{-1+\frac{\eta}{2}} \right) - g^{\mu\nu} \left(q^2 z^2 - \frac{1}{2}(4-\eta) \frac{D_2(z)}{-1+\frac{\eta}{2}} \right) \right. \\
& \left. -z q^\mu q^\nu - q^\mu q^\nu z + g^{\mu\nu} q^2 z + m^2 g^{\mu\nu} \right] \\
\times \quad & \underbrace{i\pi^{\frac{4-\eta}{2}} \frac{\Gamma\left(2-\frac{4-\eta}{2}\right)}{\Gamma(2)} \left(D_2(z)\right)^{\frac{4-\eta}{2}-2}}_{I_0}
\end{aligned}
$$

Vacuum polarization

$$
\begin{aligned}
i e_0^2 \Pi^{\mu\nu}(q) &= -i \frac{e_0^2}{4\pi^2} \frac{1}{(4\pi\mu^2)^{-\frac{\eta}{2}}} \frac{\Gamma\left(\frac{\eta}{2}\right)}{1} \\
&\int_0^1 \mathrm{d}z \left[2q^\mu q^\nu z^2 - g^{\mu\nu} \frac{D_2(z)}{-1+\frac{\eta}{2}} - g^{\mu\nu} q^2 z^2 + \frac{1}{2} g^{\mu\nu}(4-\eta) \frac{D_2(z)}{-1+\frac{\eta}{2}} \right. \\
&\qquad \left. -2q^\mu q^\nu z + g^{\mu\nu} q^2 z + m^2 g^{\mu\nu} \right] D_2(z)^{-\frac{\eta}{2}} \\
&= -i \frac{e_0^2}{4\pi^2} \left(\frac{2}{\eta} - \gamma_E \right) \left(4\pi\mu^2\right)^{\frac{\eta}{2}} \int_0^1 \mathrm{d}z \left\{ q^\mu q^\nu \left(2z^2 - 2z\right) \right. \\
&-g^{\mu\nu} \left[\left(-1 - \frac{\eta}{2} + 2 + \frac{\eta}{2} \right) \left(-q^2 z(1-z) + m^2 \right) \right. \\
&\left. \left. +q^2 z^2 - q^2 z - m^2 \right] \right\} D_2(z)^{-\frac{\eta}{2}}.
\end{aligned}
$$

Vacuum polarization

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\begin{aligned}
ie_0^2\Pi^{\mu\nu}(q) &= -i\frac{e_0^2}{4\pi^2}\left(\frac{2}{\eta}-\gamma_E\right)\int\limits_0^1 \mathrm{d}z\,\{-2q^\mu q^\nu z\,(1-z) \\
&\quad +2g^{\mu\nu}q^2 z(1-z)\}\left(\frac{D_2(z)}{4\pi\mu^2}\right)^{-\frac{\eta}{2}} \\
&= \left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\,\frac{ie_0^2}{2\pi^2}\left(\frac{2}{\eta}-\gamma_E\right)\int\limits_0^1 \mathrm{d}z z\,(1-z)\left(1-\frac{\eta}{2}\ln\frac{D_2(z)}{4\pi\mu^2}\right).
\end{aligned}
$$

We see that

$$
\Pi^{\mu\nu}(q) = \left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\Pi\left(q^2\right),
$$

Vacuum polarization

where

$$\Pi\left(q^2\right) = \frac{1}{2\pi^2}\left[\left(\frac{2}{\eta} - \gamma_E\right)\frac{1}{6} - \int_0^1 \mathrm{d}z\, z\,(1-z)\ln\frac{D_2(z)}{4\pi\mu^2}\right]$$

$$= \frac{1}{12\pi^2}\left(\frac{2}{\eta} - \gamma_E + \ln\left(4\pi\mu^2\right)\right)$$

$$-\frac{1}{2\pi^2}\int_0^1 \mathrm{d}z\, z\,(1-z)\ln\left(m^2 - i\varepsilon - q^2 z(1-z)\right).$$

Denote: C_{UV} and $I(q^2)$, then

$$\Pi\left(q^2\right) = \frac{1}{12\pi^2}\left[C_{UV} - 6I\left(q^2\right)\right].$$

Vacuum polarization

Note that

$$I(0) = \int_0^1 \mathrm{d}z\, z\,(1-z)\ln m^2 = \frac{1}{6}\ln m^2.$$

Define the finite function:

$$X(q^2) = 6I(q^2) - \ln m^2 \quad \Rightarrow \quad X(0) = 0,$$

then

$$\Pi\left(q^2\right) = \frac{1}{12\pi^2}\left[C_{UV} - \ln m^2 - X\left(q^2\right)\right].$$

Vacuum polarization

The amplitudae in the Feynman gauge has the form:

$$M = \bar{v}(p_2) i e_0 \gamma^\alpha u(p_1) \frac{-i g_{\alpha\mu}}{q^2} i e_0^2 \Pi^{\mu\nu}(q) \frac{-i g_{\nu\beta}}{q^2} \bar{u}(q_1) i e_0 \gamma^\beta v(q_2)$$

$$=$$

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\begin{aligned}
M &= \bar{v}(p_2) i e_0 \gamma^\alpha u(p_1) \frac{-i g_{\alpha\mu}}{q^2} i e_0^2 \Pi^{\mu\nu}(q) \frac{-i g_{\nu\beta}}{q^2} \bar{u}(q_1) i e_0 \gamma^\beta v(q_2) \\
&= \frac{i e_0^2}{q^2} \bar{v}(p_2) \gamma_\mu u(p_1) e_0^2 \left(q^\mu q^\nu - g^{\mu\nu} q^2 \right) \Pi \left(q^2 \right) \frac{1}{q^2} \bar{u}(q_1) \gamma_\nu v(q_2),
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&= \frac{i e_0^2}{q^2} \bar{v}(p_2) \gamma_\mu u(p_1) e_0^2 \left(q^\mu q^\nu - g^{\mu\nu} q^2 \right) \Pi \left(q^2 \right) \frac{1}{q^2} \bar{u}(q_1) \gamma_\nu v(q_2),
\end{aligned}
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$$
q^\nu \bar{u}(q_1) \gamma_\nu v(q_2) = \bar{u}(q_1) \left(\not{q}_1 + \not{q}_2 \right) v(q_2) = \bar{u}(q_1)(m - m) v(q_2) = 0,
$$

Vacuum polarization

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&= \frac{i e_0^2}{q^2} \bar{v}(p_2) \gamma_\mu u(p_1) e_0^2 \left(q^\mu q^\nu - g^{\mu\nu} q^2 \right) \Pi \left(q^2 \right) \frac{1}{q^2} \bar{u}(q_1) \gamma_\nu v(q_2),
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$$

Vacuum polarization

The amplitudae in the Feynman gauge has the form:

$$
\begin{aligned}
M &= \bar{v}(p_2) i e_0 \gamma^\alpha u(p_1) \frac{-i g_{\alpha\mu}}{q^2} i e_0^2 \Pi^{\mu\nu}(q) \frac{-i g_{\nu\beta}}{q^2} \bar{u}(q_1) i e_0 \gamma^\beta v(q_2) \\
&= \frac{i e_0^2}{q^2} \bar{v}(p_2) \gamma_\mu u(p_1) e_0^2 \left(q^\mu q^\nu - g^{\mu\nu} q^2 \right) \Pi\left(q^2\right) \frac{1}{q^2} \bar{u}(q_1) \gamma_\nu v(q_2),
\end{aligned}
$$

$$
q^\nu \bar{u}(q_1) \gamma_\nu v(q_2) = \bar{u}(q_1) \left(\not{q}_1 + \not{q}_2 \right) v(q_2) = \bar{u}(q_1)(m - m) v(q_2) = 0,
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and the sum of the Born amplitude and the one-loop correction reads:

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M_0 + M = \bar{v}(p_2) \gamma_\mu u(p_1) \frac{i e_0^2}{q^2} \left[1 - e_0^2 \Pi\left(q^2\right) \right] \bar{u}(q_1) \gamma^\mu v(q_2).
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We see that, compared to the amplitude M_0, the charge is replaced with:

$$e_0^2 \quad \longrightarrow \quad e_0^2\left[1 - e_0^2 \Pi\left(q^2\right)\right] \quad \longrightarrow \quad \frac{e_0^2}{1 + e_0^2 \Pi\left(q^2\right)}$$

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Charge renormalization

- Define the renormalized charge in the one-loop approximation:

$$e_R^2 = e_0^2\,(1+\alpha_0 C_1) \quad \Rightarrow \quad e_0^2 = e_R^2\,(1-\alpha_R C_1)\,,$$

- or exactly:

$$e_R^2 = e_0^2\left(1+\sum_{n=1}^{\infty}\alpha_0^n C_n\right) \qquad \Rightarrow \qquad e_0^2 = e_R^2\left(1+\sum_{n=1}^{\infty}\alpha_R^n B_n\right).$$

- Let us calculate

$$\begin{aligned} e_0^2\left[1-e_0^2\Pi\left(q^2\right)\right] &= e_R^2\,(1-\alpha_R C_1)\,[1+C_1\alpha_R\,(1-\alpha_R C_1) \\ &+ \frac{X(q^2)}{3\pi}\alpha_R\,(1-\alpha_R C_1)\Big] \end{aligned}$$

Charge renormalization

- The infinities contained in C_1 cancel each other

$$e_0^2\left[1-e_0^2\Pi\left(q^2\right)\right]=e_R^2\left[1+\alpha_R\left(C_1-C_1\right)+\frac{\alpha_R}{3\pi}X(q^2)\right],$$

  where we have discarded terms $\sim\alpha_R^2$ in the square brackets on the right hand side.

- The renormalized amplitude in the one-loop approximation reads:

$$M_0+M=\bar{v}(p_2)\gamma_\mu u(p_1)\frac{ie_R^2\left[1+\frac{\alpha_R}{3\pi}X(q^2)\right]}{q^2}\bar{u}(q_1)\gamma^\mu v(q_2).$$

We see that it is finite.